How to ensure that a domain decomposition method will converge.

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Abstract. The simulation of processes in highly heterogeneous media comes with many challenges. In particular many domain decomposition methods do not perform well in this case, specially if the decomposition into subdomains does not accommodate the coefficient variations. For three popular domain decomposition methods (two level additive Schwarz, BDD and FETI) we have proposed a remedy to this problem in previous work with coauthors. Here we present the strategy which was used by applying it to the Hybrid Schwarz preconditioner. It is based on identifying a bottleneck estimate in the proof of convergence which cannot be satisfied for the entire solution space. Then the part of the solution which is problematic is isolated *via* a generalized eigenvalue problem and solved separately.

Keywords: domain decomposition, robustness, heterogeneous problems, Hybrid Schwarz

1 Introduction

The method presented here is a different application of a strategy devised in [7, 8], in collaboration with Victorita Dolean, Patrice Hauret, Frédéric Nataf, Clemens Pechstein and Robert Scheichl, and generalized in [3] with Daniel J. Rixen. It is also closely related to the work of [6]. In Section 2 we present the one level Schwarz preconditioner, its two level extension based on projections (also known as hybrid Schwarz) and state clearly our objective. In Section 3 we present the theoretical analysis of this preconditioner based on [4], find which is the bottleneck estimate and put it in a local form (11). Then we define the coarse space (Definition 1) and give the main result (16): an estimate for the convergence of the solver that does not depend on the number of subdomains or the parameters in the equations. Finally in Section 4 we give a numerical illustration for two dimensional linear elasticity with highly heterogeneous coefficients.

2 Two Level Schwarz Method with Projection (aka Hybrid Schwarz)

2.1 One Level Schwarz method

Maybe the most straightforward of the domain decomposition methods is the Additive Schwarz method [4]. The information needed to build the additive Schwarz preconditioner is the following:

- A set $\omega = \{1, \dots, n\}$ of degrees of freedom,
- A set of symmetric positive semi-definite element matrices $\{A_\tau \in \mathbb{R}^{n\times n}; \tau \in \mathcal{T}_h\}$, which give the weights of the connections between degrees of freedom,
- The connectivity graph for each connection $\tau \in \mathcal{T}_h$ which is the list $dof(\tau) \subset \omega$ of degrees of freedom which are connected to others through τ.

If the problem stems from the finite element approximation of a partial differential equation, these have geometrical interpretations: $\mathcal{T}_h$ is the mesh of the global domain, τ is an element of this mesh and $dof(\tau)$ is the set of degrees of freedom attached to the vertices of τ.

The global problem matrix is assembled as: $A := \sum_{\tau\in\mathcal{T}_h} A_\tau$. We suppose that A is symmetric positive definite (spd). Then, given a right hand side $f \in \mathbb{R}^n$ the objective is to solve:

$$\text{Find } x^* \in \mathbb{R}^n \text{ such that } Ax^* = f. \tag{1}$$

The idea behind the Additive Schwarz preconditioner is to approximate the global inverse of A by a sum of local inverse A_j^{-1}. The local inverses are based on an overlapping partition of the set of degrees of freedom ω:

$$\omega = \omega_1 \cup \ldots \cup \omega_N, \text{ such that } \forall m \in \omega, \exists j = 1, \dots, N; \left(\bigcup_{\{\tau, m\in dof(\tau)\}} dof(\tau) \right) \subset \omega_j.$$

This condition says that each degree of freedom m is in the interior of at least one subdomain ω_j in the sense that its neighbours, $\bigcup_{\{\tau, m\in dof(\tau)\}} dof(\tau)$, are also in this subdomain. The convergence of the additive Schwarz method relies on this property. The final ingredient for building the one level additive Schwarz preconditioner is a set of interpolation matrices between the global unknowns and the local unknowns. For any $j = 1, \dots, N$ let n_j be the cardinality of ω_j, then the restriction matrix $R_j \in \mathbb{R}^{n_j\times n}$ is the boolean matrix with one 1 entry on each line which corresponds to a degree of freedom in ω_j. With this the one level Schwarz preconditioner writes:

$$M^{-1} := \sum_{j=1}^{N} R_j^\top A_j^{-1} R_j, \quad A_j := R_j A R_j^\top. \tag{2}$$

The matrices A_j are built by extracting the coefficients in the global matrix A which correspond to degrees of freedom in ω_j. For this reason they are also symmetric positive definite. Unfortunately, the conjugate gradient algorithm preconditioned with the one level Additive Schwarz method is usually not scalable. This means that its convergence rate strongly depends on the number of subdomains. Another drawback is that the convergence rate deteriorates when there are jumps in the coefficients of the underlying set of partial differential equations. One way to improve this is to use a projected operator.

2.2 Adding Projection Steps

In [1] the idea to use a projection as a preconditioner for a Krylov method is introduced. Having chosen a set V_0 of vectors in $\mathbb{R}^n$ which are spanned by the n_0 (linearly independ) columns of $R_0^\top \in \mathbb{R}^{n\times \# V_0}$, we build the A-orthogonal projection operator onto V_0:

$$P_0 := R_0^\top (R_0 A R_0^\top)^{-1} R_0 A. \tag{3}$$

Using the A-orthogonality of P_0, the original problem (1) rewrites as two idependant problems: Find $x^* \in \mathbb{R}^n$ such that

$$(I-P_0)^\top A(I-P_0)x^* = (I-P_0^\top)f, \text{ and } P_0^\top A P_0 x^* = P_0^\top f. \tag{4}$$

The number of vectors in V_0 is supposed to be sufficiently small so that we can solve the second equation with a direct solve to get $P_0 x^*$. We then apply the conjugate gradient iterations only to the first equation. The rationale behind this splitting of the solution is that even if A is ill-conditioned, in many cases the ill-conditioning is caused only by a small number of *smooth* vectors. If we can identify these vectors and use them to span the projection space V_0 then we are left with solving iteratively a problem for a much better conditioned matrix $(I-P_0)^\top A(I-P_0)$.

In our case we use both preconditioning and projection. The projected and preconditioned problem is to find $x^* \in \mathbb{R}^n$ such that

$$M^{-1}(I-P_0)^\top A(I-P_0)x^* = M^{-1}(I-P_0^\top)f, \text{ and } P_0^\top A P_0 x^* = P_0^\top f. \tag{5}$$

Our objective is the following: we want to design a solver which is scalable and which is robust with respect to heterogeneous coefficients in the set of underlying partial differential equations. We know an estimate for the convergence rate of the preconditioned conjugate gradient method which depends only on the condition number of the preconditioned operator (see [2] for instance). This is why the criterion for choosing our projection space is:

> Identify a space V_0 which is sufficiently small for $P_0^\top A P_0$ to be inverted using a direct solver and such that the condition number of $M^{-1}(I-P_0)^\top A(I-P_0)$ does not depend on the number of subdomains N or on any of the parameters in the original set of equations.

Intuitively, the vectors which are used for the projection space are the parts of the solution for which the preconditioner does not do a good job ($M^{-1}Ax$ is very different from x).

3 Choosing the Projection Space

3.1 Abstract Schwarz Theory

Fortunately the additive Schwarz preconditioner has already been thoroughly analyzed and an abstract presentation of this theory can be found in [4] (see hybrid Schwarz). Our choice for the projection space relies on this analysis.

The iterations of the preconditioned conjugate gradient for operator $(I - P_0)^\top A(I-P_0)$ and preconditioner M^{-1} produce the same values as the iterations of the conjugate gradient for the symmetric operator $M^{-1/2}(I - P_0)^\top A(I - P_0)M^{-1/2}$ where the right hand side f is replaced with $M^{-1/2}f$ and the unknown is $M^{1/2}x$ [2]. We will obtain estimates for the condition number by finding an upper bound for the, respectively, largest and lowest eigenvalues λ_{max} and λ_{min} of $M^{-1/2}(I - P_0)^\top A(I - P_0)M^{-1/2}$ in $\mathrm{range}(M^{1/2}(I - P_0))$.

Notice that the reformulation does not contradict our previous objective since the eigenvalues and thus the condition number of $M^{-1/2}(I - P_0)^\top A(I - P_0)M^{-1/2}$ in $\mathrm{range}(M^{1/2}(I - P_0))$ are the same as the eigenvalues of $M^{-1}A$ in $\mathrm{range}(I - P_0)$.

Let $y \in \mathrm{range}(M^{1/2}(I - P_0))$, $y = M^{1/2}x$, $x \in \mathrm{range}(I - P_0)$, then

$$\langle M^{-1/2}(I - P_0)^\top A(I - P_0)M^{-1/2}y, y\rangle = \langle Ax, x\rangle \text{ and } \langle y, y\rangle = \langle Mx, x\rangle \tag{6}$$

so if there exist constants C_1 and C_2 such that

$$C_1\langle Mx, x\rangle \le \langle Ax, x\rangle \le C_2\langle Mx, x\rangle,\ \forall\, x \in \mathrm{range}(I - P_0) \tag{7}$$

then $\lambda_{max} \le C_2$, $\lambda_{min} \ge C_1$ and the condition number is bounded by C_2/C_1. We notice that the constants which we want to evaluate measure, on $\mathrm{range}(I - P_0)$, the difference between the energy norm with respect to the original operator A and the energy norm with respect to the inverse of the preconditioner M (this is the inverse of an approximation of the inverse of A). Since we have fixed the choice of the preconditioner, the lattitude we are left with in order to satisfy the estimate is to put the vectors for which we cannot write the proof into the projection space V_0.

In practice we never need to compute the inverse M of the preconditioner. In our analysis we will use the expression given in Lemma 2.5 of [4][1]:

$$\langle Mx, x\rangle = \min_{\{x_j \in \mathbb{R}^{n_j};\, x = \sum_{j=1}^N R_j^\top x_j\}} \sum_{j=1}^N \langle A_j x_j, x_j\rangle. \tag{8}$$

The energy norm of x with respect to the inverse of the preconditioner M minimizes the sum, over all the possible decompositions of x onto the N subdomains, of the local energies.

Next we prove an estimate for λ_{max}. This estimate depends on the maximal number $\mathcal{N}^c$ of colors that are needed to color each of the sets ω_j in such a

[1] In the book $P_{ad} = M^{-1}A$ so $AP_{ad}^{-1} = M$.

way that two subsets with the same color are A-orthogonal. More precisely let $color(j) \in \{1, \ldots, \mathcal{N}^c\}$ denote the color of a subdomain j then

$$\langle AR_k^\top u_k, R_l^\top u_l\rangle = 0, \, \forall u_k \in \omega_k \text{ and } u_l \in \omega_l \text{ if } color(k) = color(l).$$

Given the decomposition $x = \sum_{j=1}^N R_j^\top x_j$ which realizes the minimum in (8) we write

$$\begin{aligned} \langle Mx, x\rangle &= \textstyle\sum_{j=1}^N \langle AR_j^\top x_j, R_j^\top x_j\rangle \\ &= \textstyle\sum_{c=1}^{\mathcal{N}^c} \langle A \sum\limits_{\{i;color(i)=c\}} R_i^\top x_i, \sum\limits_{\{i;color(i)=c\}} R_i^\top x_i\rangle \\ &\geq \tfrac{1}{\mathcal{N}^c} \langle A \sum_{j=1}^N R_j^\top x_j, \sum_{j=1}^N R_j^\top x_j\rangle \\ &= \tfrac{1}{\mathcal{N}^c} \langle Ax, x\rangle. \end{aligned} \tag{9}$$

The argument for the inequality in the third line is a generalization of the identity $2(a^2 + b^2) \geq (a + b)^2$. We have proved that $\lambda_{max} \leq \mathcal{N}^c$. This estimate does not depend on the number of subdomains or the coefficients in the equations and it holds independently of the choice of the projection space.

Deriving a bound for λ_{min} is a trickier job.

3.2 Identifying the Bottleneck

We procede by beginning to write the proof for an arbitrary contant C. We will find a sufficient condition for this bound to be true which has the nice feature of being local. Then we will build the projection space by solving a generalized eigenvalue problem which identifies which parts of the local subspace satisfy the sufficient condition or not. Those that do not will serve as a basis for the projection space.

$$\begin{aligned} &\langle Ax, x\rangle \geq C\langle Mx, x\rangle \\ \Leftrightarrow\ &\langle Ax, x\rangle \geq C \min_{\{x_j \in \mathbb{R}^{n_j}; x = \sum_{j=1}^N R_j^\top x_j\}} \textstyle\sum_{j=1}^N \langle A_j x_j, x_j\rangle \\ \Leftrightarrow\ &\langle Ax, x\rangle \geq C \min_{\{x_j \in \mathbb{R}^{n_j}; x = \sum_{j=1}^N R_j^\top x_j\}} \textstyle\sum_{j=1}^N \langle A(I - P_0)R_j^\top x_j, (I - P_0)R_j^\top x_j\rangle. \end{aligned} \tag{10}$$

The last equivalence is too long to prove here. The idea is to look at the projected Additive Schwarz preconditioner as the textbook additive Schwarz preconditioner for the projected operator $(I-P_0)^\top A(I-P_0)$ with prolongation operators $R_j^\top$ replaced by $(I-P_0)R_j^\top$. A sufficient condition for (10) to be true is that this inequality hold for one particular choice of the decomposition of x so we choose one. Let $D_j \in \mathbb{R}^{n_j \times n_j}$ be diagonal weghting matrices which form a partition of unity: $\sum_{j=1}^N R_j^\top D_j R_j = I$ (I is the identity in $\mathbb{R}^{n \times n}$). Then let $x_j = D_j R_j x$. We have built a decomposition of x onto the subspaces ($\sum_{j=1}^N R_j^\top x_j = x$) and

$$\langle Ax, x\rangle \geq C \sum_{j=1}^N \langle A(I - P_0)R_j^\top D_j R_j x, (I - P_0)R_j^\top D_j R_j x\rangle \Rightarrow (10).$$

The final step to make the condition local is to make the left hand side local. We recall that A is assembled as a sum of element matrices $A = \sum_{\tau\in\mathcal{T}_h} A_\tau$. Each of these element matrices was supposed to be symmetric positive semi definite. This means that if we assemble the element matrices over a subset $\mathcal{T}_h^j$ of $\mathcal{T}_h$ the resulting energy norm will be bounded with respect to $\langle A\cdot,\cdot\rangle^{1/2}$. In particular, let $\mathcal{T}_h^j = \{\tau; dof(\tau)\subset\omega_j\}$ be the set of connections which are completely in subdomain j and define the corresponding local matrix as $\tilde{A}_j = \sum_{\tau\in\mathcal{T}_h^j} R_j^\top A_\tau R_j$ then $\sum_{j=1}^N \langle \tilde{A}_j R_j x, R_j x\rangle \le \mathcal{N}^c\langle Ax, x\rangle$, and the sufficient condition becomes local:

$$\langle \tilde{A}_j R_j x, R_j x\rangle \ge \frac{C}{\mathcal{N}^c}\langle A(I-P_0)R_j^\top D_j R_j x, (I-P_0)R_j^\top D_j R_j x\rangle \Rightarrow (10). \tag{11}$$

We do not know how to simplify this condition further without using information on the underlying set of partial differential equations and on the partition of unity so this is the bottleneck estimate which we have used to select which vectors span the projection space. Next we give its definition:

3.3 Building the Projection Space to satisfy the Bottleneck

Definition 1. *For any subdomain $j = 1,\ldots,N$, find the eigenpairs $(p_j^k, \lambda_j^k) \in \mathbb{R}^{n_j}\times\mathbb{R}^+$ of*

$$\tilde{A}_j p_j^k = \lambda_j^k D_j A_j D_j p_j^k. \tag{12}$$

Then for a given threshold $\mathcal{K}$ let the coarse space be defined by

$$V_0 = \bigcup_{j=1,\ldots,N} R_j^\top D_j(V_0^j);\; V_0^j = \mathit{span}\{p_j^k; \lambda_j^k < \mathcal{K}\}.$$

Suppose that the eigenvectors have been normalized so that $\langle D_j A_j D_j p_j^k, p_j^k\rangle = 1$. Maybe the most important arguments for writing the proof are

$$\langle A_j D_j p_j^k, D_j p_j^l\rangle = 0 \text{ and } \langle \tilde{A}_j p_j^k, p_j^l\rangle = 0,\ \text{ if } k\neq l.$$

We may use the vectors $R_j^\top D_j p_j^k \in V_0$ as columns for the interpolation operator $R_0^\top$ from V_0 into $\mathbb{R}^n$. Then we build the projector P_0 following (3). This way we have completed the definition of our domain decomposition method. All that is left to do is to make sure that the projection space introduced in Definition 1 does indeed do its job and that the bottleneck estimate (11) holds for any $x \in$ range$(I-P_0)$.

If P_0^j is the A-orthogonal projection onto the set $\{R_j^\top D_j p_j^k; \lambda_j^k < \mathcal{K}_j\}$ then

$$\langle A(I-P_0)u, (I-P_0)u\rangle \le \langle A(I-P_0^j)u, (I-P_0^j)u\rangle,\ \forall u\in\omega. \tag{13}$$

Moreover, if $\Pi_j : \omega_j \to \omega_j$, $\Pi_j x_j = \sum_{\{k;\lambda_j^k<\mathcal{K}\}}\langle D_j A_j D_j x_j, p_j^k\rangle p_j^k$ is the projection operator from [8] then $P_0^j R_j^\top D_j = R_j^\top D_j \Pi_j$ so

$$\langle A(I-P_0^j)R_j^\top D_j R_j x, (I-P_0)R_j^\top D_j R_j x\rangle = \langle D_j A_j D_j (I-\Pi_j)R_j x, (I-\Pi_j)R_j x\rangle. \tag{14}$$

We apply the abstract Lemma 2.11 from [8] and then use the fact that $\langle \tilde{A}_j p_j^k, p_j^l\rangle = 0$, $k \neq l$, to get

$$\begin{aligned} \langle D_j A_j D_j (I - \Pi_j) R_j x, (I - \Pi_j) R_j x\rangle &\leq \tfrac{1}{\mathcal{K}} \langle \tilde{A}_j (I - \Pi_j) R_j x, (I - \Pi_j) R_j x\rangle \\ &\leq \tfrac{1}{\mathcal{K}} \langle \tilde{A}_j R_j x, R_j x\rangle . \end{aligned} \tag{15}$$

Putting (15) together with (13) and (14) proves the condition in (11) for $C/\mathcal{N}^c = \mathcal{K}$ so $\lambda_{min} \geq \mathcal{K}/\mathcal{N}^c$ and the condition number is bounded by ${\mathcal{N}^c}^2/\mathcal{K}$. Hence if x^* is the exact solution of the original problem (1), x^0 is the initial guess, and x^m is the approximate solution given by the m-th step of the preconditioned conjugate gradient algorithm with the projected Additive Schwarz preconditioner, the error decreases at least as

$$\frac{\|x^* - x^m\|_A}{\|x^* - x^0\|_A} \leq 2 \left[\frac{\sqrt{\mathcal{K}} - \mathcal{N}^c}{\sqrt{\mathcal{K}} + \mathcal{N}^c} \right]^m , \tag{16}$$

where $\| \cdot \|_A = \langle A \cdot, \cdot \rangle^{1/2}$, $\mathcal{K}$ is the **chosen** threshold used to select eigenvectors for the projection space in Definition 1, and $\mathcal{N}^c$ is the number of colors that are needed to color the subdomains in such a way that two subdomains with the same color are orthogonal.

4 Numerical Illustration

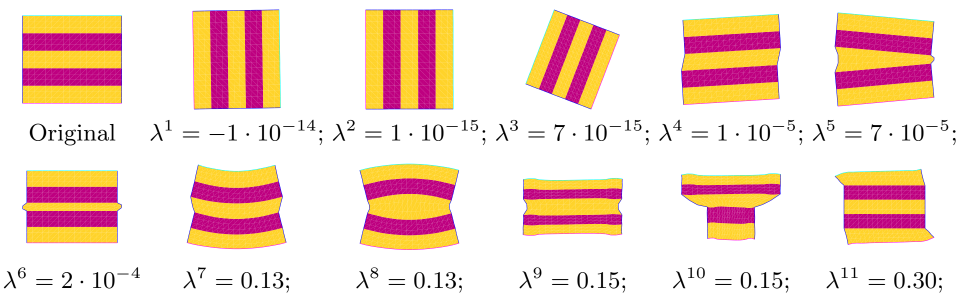

Fig. 1. Original configuration and first eleven eigenvectors for a floating subdomain. With $\mathcal{K} = 0.1$ we select six eigenvectors for the projection space. Among these, the first three correspond to the rigid body modes ($\tilde{A}_j p_j^{1,2,3} = 0$). In total the size of the projection space is 46.

In this section for lack of space we have chosen to illustrate the way that the method works rather than a set of performance tests. The implementation uses matlab and Freefem++. We solve the two dimensional linear elasticity equations discretized with $\mathbb{P}_1$ (piecewise linear) finite elements on a 121×16 regular mesh with simplicial elements. The domain is an 8×1 rectangle which we decompose

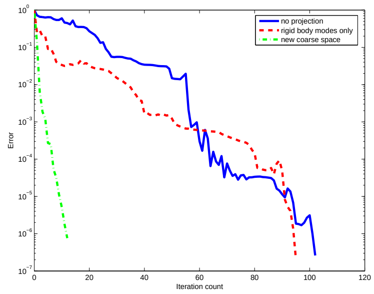

Fig. 2. Error versus the iteration count for three methods: no projection (blue full line), projection onto the rigid body modes (red hashes), projection onto the space from definition 1 (green hashes and dots). The new projections space does its job. The condition number is reduced from 3576 to 13. With just the rigid body modes it is 1808.

into 8 side by side unit squares. Then we add one layer of overlap to each. The medium is a soft material (Lamé parameters: $E = 10^7$, $\nu = 0.4$) with two layers of a harder material ($E = 10^{12}$, $\nu = 0.4$). In Figure 4 we have plotted the original configuration for one subdomain as well as the first eigenmodes for eigenproblem (12). In Figure 4 we show that the new method converges very fast and that the projection step does its job since it reduces the condition number from 3576 to 13 using only 46 projection vectors.

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